MATH 2050C Lecture 17 (Mar 17)

Problem Set 9 posted and due on Mar 25.

Last time: "Sequential Criteria" $\boxed{f: A \to \mathbb{R}}$

$\lim_{x \to c} f(x) = L \iff \forall$ seq. (x_n) in A s.t. $x_n \neq c \ \forall n \in \mathbb{N}$, $\lim(x_n) = c$ we have $(f(x_n)) \to L$.

Applications:
- divergence criteria
- "limit theorems" for functions

Limit Theorems for functions (§ 4.2 in Bartle's)

Recall: (x_n) convergent $\Rightarrow$ (x_n) bdd.

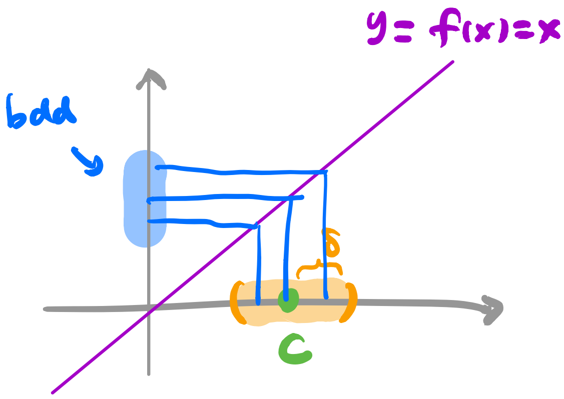
Boundedness Thm:

$\lim_{x \to c} f(x)$ exists $\Rightarrow$ f is "bdd in a neighborhood of c"

i.e. $\exists\, M > 0$ and $\exists\, \delta > 0$ s.t.

$|f(x)| \leq M \quad \forall x \in A$ s.t. $|x - c| < \delta$

E.g.) $f(x) = x$



Remark: f may not be bdd "globally".

Proof: By def$^{\underline{n}}$, $\lim_{x\to c} f(x) = L \Rightarrow$ take $\varepsilon = 1$,

$\exists\ \delta = \delta(1) > 0$ s.t $|f(x) - L| < 1$

whenever $x \in A$, $0 < |x - c| < \delta$

Then, by Δ-ineq., this implies

$$|f(x)| \leq |f(x) - L| + |L| < 1 + |L|$$

whenever $x \in A$, $0 < |x - c| < \delta$

If we take $M := \max\{1 + |L|, \underbrace{|f(c)|}_{\text{if } c \in A}\} > 0$

then $|f(x)| \leq M \quad \forall\, x \in A,\ |x - c| < \delta$.

$\square$

Def$^{\underline{n}}$: Given $f, g : A \to \mathbb{R}$ functions, we can define some new functions as follow:

- $(f \pm g)(x) := f(x) \pm g(x)$, $\quad f \pm g : A \to \mathbb{R}$
- $(fg)(x) := f(x) \cdot g(x)$, $\quad fg : A \to \mathbb{R}$
- $\left(\frac{f}{g}\right)(x) := \dfrac{f(x)}{g(x)}$, $\quad \frac{f}{g} : A \setminus \{x \in A \mid g(x) = 0\} \to \mathbb{R}$

Thm: (1) $\lim\limits_{x\to c}(f\pm g)(x) = \lim\limits_{x\to c} f(x) \pm \lim\limits_{x\to c} g(x)$

(2) $\lim\limits_{x\to c}(fg)(x) = \lim\limits_{x\to c} f(x) \cdot \lim\limits_{x\to c} g(x)$

(3) $\lim\limits_{x\to c}\left(\frac{f}{g}\right)(x) = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}$

provided that $\lim\limits_{x\to c} f(x)$, $\lim\limits_{x\to c} g(x)$ exist.

(and for (3), additionally, $\lim\limits_{x\to c} g(x) \neq 0$)

Examples: $\lim\limits_{x\to c}\frac{1}{x} = \frac{1}{c}$, $c\neq 0$

$\lim\limits_{x\to 2}\dfrac{x^3-4}{x+1} = \dfrac{4}{3}$; $\lim\limits_{x\to 2}\dfrac{x^2-4}{3x-6} = \lim\limits_{x\to 2}\dfrac{x+2}{3} = \dfrac{4}{3}$.

Proof of (2): Idea: use seq. criteria.

Take an arbitrary seq. (x_n) in A s.t

$x_n \neq c$ $\forall n\in\mathbb{N}$ and $\lim(x_n) = c$.

Seq. Criteria $\Rightarrow$ $(f(x_n)) \to \lim\limits_{x\to c} f(x)$; $(g(x_n)) \to \lim\limits_{x\to c} g(x)$

Limit Thm for seq $\Rightarrow$ $(f(x_n)\cdot g(x_n)) \to \lim\limits_{x\to c} f(x)\cdot \lim\limits_{x\to c} g(x)$

Seq criteria $\Rightarrow$ $\lim\limits_{x\to c}(fg)(x) = \lim\limits_{x\to c} f(x)\cdot\lim\limits_{x\to c} g(x)$

$\blacksquare$

Squeeze / Sandwich Thm:

Let $g, f, h : A \to \mathbb{R}$ be functions s.t

$$\boxed{g(x) \le f(x) \le h(x) \quad \forall x \in A} \quad \cdots (\dagger)$$

Suppose $\lim_{x \to c} g(x) = L = \lim_{x \to c} h(x)$.

THEN, $\lim_{x \to c} f(x) = L$

Remarks: (1) We do not need to assume that $\lim_{x \to c} f(x)$ exists. it follows as a conclusion.

(2) One only needs $(\dagger)$ to hold "locally" in a neighborhood of c.

Proof: Take any arbitrary seq (x_n) in A s.t $x_n \neq c$ $\forall n \in \mathbb{N}$, $\lim (x_n) = c$

By $(\dagger)$, $g(x_n) \le f(x_n) \le h(x_n)$ $\forall n \in \mathbb{N}$

Seq Criteria $\Rightarrow$ $\lim (g(x_n)) = L = \lim (h(x_n))$

Squeeze Thm for seq. $\Rightarrow$ $\lim (f(x_n)) = L$

Seq Criteria $\Rightarrow$ $\lim_{x \to c} f(x) = L$

$\square$